Electromagnetic Fields in a Homogeneous, Nonisotropic Universe*

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A solution of the Einstein-Maxwell equations is derived which represents a closed universe of topology $S^3 \times R$, filled with gravitational and electromagnetic radiation. We confine attention to the lowest of the large number of possible modes of radiation in such a universe. This mode has maximum symmetry consistent with the existence of a vector field; the universe is homogeneous but not isotropic, and is therefore a generalization of one of the solutions discussed by Taub. It is possible to solve explicitly for the metric coefficients. Some of the physical properties of the solution are discussed.

I. INTRODUCTION

IN 1951, A. H. Taub[1] gave a two-parameter family of solutions to the sourceless Einstein equations ($R_{\mu\nu}=0$) which are characterized by their symmetry under $O(3)$, the rotation group of the 3-sphere. Only later was it realized[2–6] that for one region of the coordinates this solution describes a closed space evolving in time ("Taub universe") and that the analytic extension to other regions of the coordinates describes asymptotically flat spaces ("outer NUT space"). Closed spaces without sources have sometimes been proposed as anti-Mach universes[7]; alternatively, we here adopt the view that the Taub universe is held together by its content of *gravitational radiation*, which is present in the lowest possible mode (maximum symmetry). In order better to understand such radiation in the Taub universe, we give a generalization of the Taub-NUT solution which allows for presence of electromagnetic radiation of the same maximum symmetry.

To date the only well-known example of a universe held together by its radiation content is the Tolman universe. In this solution of Einstein's equations the wavelength λ of the radiation is very small compared to the radius of the universe, and the radiation is distributed over many modes. Therefore a statistical treatment of the radiation is appropriate, and the universe may be assumed homogeneous and isotropic in the large. Thus the Tolman universe is a limiting case ($\lambda \rightarrow 0$) of a large class of solutions in which the radiation is present in various modes of finite wavelength. The spectrum of possible wavelength modes for radiation in a spherical, or topologically spherical, universe is discrete.[8] The fewer modes excited, or the longer their wavelength, the less applicable becomes the approximation of the Tolman universe and its assumption of homogeneity and isotropy. It is entirely conceivable that a universe be held together and curved into a closed space by a sufficiently strong excitation of a *single* radiation mode. The present paper investigates this case for the mode of longest possible wavelength.

For vector fields (electromagnetic radiation) and tensor fields (gravitational radiation) the lowest mode corresponds to *constant* fields on spacelike hypersurfaces $t=\text{const}$, in this sense: Each point on a hypersurface can be mapped into every other point such that metric and field direction are preserved. Following Heckmann and Schücking[3] and the common usage in mathematics[9] we call such a universe *homogeneous*. However, at any particular point not all *directions* are equivalent—the universe is not isotropic. In Taub's solution and the generalization to be considered here only *one* direction at each point is distinguished so that an additional symmetry exists, namely, rotations about the distinguished direction. Our task is to solve the sourceless Maxwell-Einstein equations in this homogeneous, nonisotropic universe.

As an example of a homogeneous space, consider the unit 3-sphere $x^2+y^2+z^2+w^2=1$, described here imbedded in Euclidean 4-space. A preferred direction is the continuous unit tangent vector field $(y, -x, z, -w)$; a typical mapping showing the homogeneity moves every point a constant distance along the field lines of this vector field. One of the solutions given below has exactly this geometry at the time $t=0$, with the electromagnetic field pointing along the preferred direction. The other solutions differ from this example only in this respect, that the 3-sphere $t=\text{const}$ is distorted so that not only the electromagnetic field, but also the space geometry singles out a preferred direction.

Since there is to be only *one* preferred direction in our solutions, both the **E** and **H** fields must point along this direction. They must therefore be parallel, and have

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[7] I. Ozsvath and E. Schücking, *Recent Developments in General Relativity* (Pergamon Press, New York, 1962), p. 339.

[8] For a spherical universe, see E. Schrödinger, Comment. Pont. Acad. **2**, 321 (1938); or *Expanding Universes* (Cambridge University Press, New York, 1956), Chap. III. For the general case, see K. Kodaira and D. C. Spencer, Ann. Math. **71**, 43 (1960).

[9] See, for example, S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press Inc., New York, 1962).

constant magnitude in space because of the homogeneity. In Sec. II we confine attention to the response of such fields to an arbitrary variation of the metric in time. As in the Tolman universe, the field strengths increase as the universe contracts, and we obtain the adiabatic law for the variation in time of the energy density in a Taub space [Eq. (15)]. The Maxwell tensor due to parallel **E** and **H** fields corresponds to "positive pressure" in directions transverse to the fields, and "negative pressure" along the field direction. In analyzing the response of the geometry to the fields we should therefore expect a corresponding behavior of the metric coefficients, e.g., after the time of time-symmetry the universe should expand in the transverse directions, and contract in the longitudinal direction. Details and a discussion of this analysis follow in Secs. III and IV.

II. ELECTROMAGNETIC FIELD IN A GIVEN HOMOGENEOUS GEOMETRY

Let the spacelike invariant varieties under the homogeneity transformations be surfaces $t=$constant. Since they are geodesically parallel we can choose time-orthogonal geodesic coordinates such that the metric takes the form,

$$ds^2=-dt^2+d\sigma^2. \tag{1}$$

Here $d\sigma^2$ is a homogeneous three-dimensional metric. It is useful to describe it in terms of an orthogonal triad, σ_x, σ_y, σ_z, which is to be invariant under homogeneity transformations. Let σ_z point along the distinguished direction, then

$$d\sigma^2=A^2\sigma_z{}^2+B^2(\sigma_x{}^2+\sigma_y{}^2)\,. \tag{2}$$

Due to homogeneity, A and B are functions of t only. The invariance group of the Taub geometry has the structure[1,3] $O(3)$. The σ's are invariant differential forms and therefore satisfy[10]

$$d\sigma_z=\sigma_x\wedge\sigma_y \text{ and cylically.} \tag{3}$$

(In holonomic coordinates this metric could be written, e.g., as

$$d\sigma^2=A^2(d\psi+\cos\theta d\varphi)^2+B^2(d\theta^2+\sin^2\theta d\varphi^2)\,,$$
$$0\leqslant\psi<4\pi\,,\quad 0\leqslant\theta<\pi\,,\quad 0\leqslant\varphi<2\pi.) \tag{4}$$

Since σ_z is the preferred direction, the **E** and **H** vectors must point along it. Therefore the Maxwell field has the form, written in the tetrad dt, σ_y, σ_y, σ_z,

$$\begin{aligned}F_{tz}&=-F_{zt}=EA\\ F_{xy}&=-F_{yx}=HB^2\end{aligned} \tag{5}$$

and all other components vanish. Here the factors were chosen so that E and H are the fields measured in an orthonormal frame,

$$\omega^0=dt\,,\quad \omega^1=B\sigma_x\,,\quad \omega^2=B\sigma_y\,,\quad \omega^3=A\sigma_z. \tag{6}$$

In Cartan's notation,[11] then

$$\begin{aligned}F&=-E\omega^0\wedge\omega^3+H\omega^1\wedge\omega^2\\ &=-EAdt\wedge\sigma_z+HB^2\sigma_x\wedge\sigma_y.\end{aligned} \tag{7}$$

Due to homogeneity, E and H are functions of t only, so that, for example, $dE=\dot{E}dt$ (here a dot denotes differentiation with respect to t). By using (7) and (3), one set of Maxwell's equations,

$$dF=0 \tag{8}$$

reduces to the single equation,

$$d(HB^2)/dt=-EA\,. \tag{9}$$

The other equations from this set are automatically fulfilled by our choice (7) of the form of F. For example, since the preferred direction σ_z has the topogolgy of the vector field mentioned in the introduction and therefore points along *closed* geodesics, the equations div**E**$=0$ and div**B**$=0$ are automatically fulfilled.

To evaluate the other Maxwell equations we need the dual *F, which follows immediately from (7):

$$\begin{aligned}{}^*F&=H\omega^0\wedge\omega^3+E\omega^1\wedge\omega^2\\ &=HAdt\wedge\sigma_z+EB^2\sigma_x\wedge\sigma_y.\end{aligned} \tag{10}$$

Thus the other set of Maxwell equations, $d^*F=0$, give

$$d(EB^2)/dt=HA\,. \tag{11}$$

Introduce new measures for the fields and the time coordinate,

$$\begin{aligned}&E'=EB^2\quad H'=HB^2\\ &dt'=(A/B^2)dt.\end{aligned} \tag{12}$$

Then the Maxwell equations (9), (11) take the simple form

$$dH'/dt'=-E'\quad dE'/dt'=H' \tag{13}$$

with the solution

$$E'=\phi\sin t'\quad H'=\phi\cos t'. \tag{14}$$

Here ϕ is a constant of integration which is related to the electromagnetic energy density in the orthonormal frame

$$T_{00}=\tfrac{1}{2}(E^2+H^2)=\phi^2/2B^4. \tag{15}$$

Equation (14) is the complete solution for the fields in a given metric of the Taub type. Although only one constant of integration appears explicitly, another is contained in the arbitrary origin of t', which is left free by Eq. (12). The two constants of integration are determined, e.g., by the initial magnitudes of **E** and **H**.

The analog of the adiabatic law for the electromagnetic energy in the Tolman universe, $T_{00}\propto$ (radius of universe)$^{-4}$, is given by Eq. (15). As in the Tolman case, Eq. (15) can be derived by a simple physical

[10] See, for example, C. Chevalley, *The Theory of Lie Groups* (Princeton University Press, Princeton, New Jersey, 1946), pp. 152 ff.

[11] E. Cartan, *Les systemes differentiels et leur applications geometriques* (Hermann et Cie., Paris, 1945); G. de Rham, *Varietés differentiables* (Hermann et Cie., Paris, 1955).

argument. For example, as A increases, work is being done *on* the fields, hence the total energy in the universe increases as A; the total volume also increases as A, so that the energy *density* T_{00} shows no dependence on A. The factor B^{-2} can be similarly explained.

III. SOLUTION OF THE MAXWELL-EINSTEIN EQUATIONS

We now turn to the Einstein equations for the case of sourceless electromagnetic radiation ($T=-R=0$),

$$R_{\mu\nu}=T_{\mu\nu}. \tag{16}$$

Here the Maxwell stress-energy tensor

$$T_{\mu\nu}=F_{\mu\alpha}F^{\alpha}{}_{\nu}-\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

should be expressed in terms of the solution for F found in Sec. III, and the resulting equations (16) solved for the metric coefficients A and B. For the case of parallel $\mathbf{E}$ and $\mathbf{H}$ fields pointing in the z direction, all off-diagonal components of $T_{\mu\nu}$ vanish, and the diagonal components can all be expressed in terms of $T_0{}^0$, given by Eq. (15):

$$T_0{}^0=T_z{}^z=-T_x{}^x=-T_y{}^y.$$

Similarly, due to the homogeneous form of the metric (2), the off-diagonal components of $R_{\mu\nu}$ vanish. Moreover, since the x and y directions are equivalent, $R_x{}^x=R_y{}^y$. Thus only three of the equations (16) are not automatically satisfied. It is convenient to divide these into a set of two equations specifying the algebraic structure of $R_{\mu\nu}$,

$$R_0{}^0=R_z{}^z=-R_x{}^x \tag{17}$$

and an equation specifying the value of, say, $R_0{}^0$:

$$R_{00}=T_{00}. \tag{18}$$

Equation (17) can be solved without specifying $\mathbf{E}$ and $\mathbf{H}$. The components of $R_\mu{}^\nu$ are easily calculated[6]

$$\begin{aligned}R_0{}^0&=2d(\dot{B}/B)/dt+2(\dot{B}/B)^2+d(\dot{A}/A)/dt+(\dot{A}/A)^2,\\ R_x{}^x&=R_y{}^y=d(\dot{B}/B)/dt+2(\dot{B}/B)^2\\&\quad+(\dot{A}\dot{B}/AB)+(2B^2-A^2)/2B^4,\\ R_z{}^z&=d(\dot{A}/A)/dt)+(\dot{A}/A)^2+2(\dot{A}\dot{B}/AB)+(A^2/2B^4).\end{aligned} \tag{19}$$

To solve the second equation (17), $R_z{}^z=-R_x{}^x$, introduce a new time coordinate,

$$dt''=(A/2B_0)dt, \tag{20}$$

where $B_0=B(t_0)$ is the value of B at some fixed time t_0. The equation then takes the form

$$d^2B/dt''^2=B_0{}^2/B^3$$

with the general solution

$$B^2=B_0{}^2+(t''-t_0'')^2. \tag{21}$$

Here B_0 and t_0'' are two constants of integration. In the following we shall assume, without loss of generality, that t_0'' vanishes.

By using (21) we can rewrite the first equation (17) in the form

$$\frac{d^2}{dt''^2}[(A^2+4B_0{}^2)(B_0{}^2+t''^2)]=0$$

with the general solution

$$A^2=-4B_0{}^2+\frac{B_0{}^2(A_0{}^2+4B_0{}^2)+Ct''}{B_0{}^2+t''^2}. \tag{22}$$

Here $A_0=A(t_0)$ and C are two constants of integration.

Finally we must check what restrictions, if any, Eq. (18) imposes. We substitute (15), (21), and (22) and find that Eq. (18) reduces to an algebraic condition

$$4B_0{}^2-A_0{}^2=2\phi^2 \tag{23}$$

relating various constants of integration.

Equations (21) and (22) give the time dependence of A and B, the coefficients determining distances in the longitudinal and transverse directions. The result is consistent with the qualitative discussion in the Introduction.

To compare with the Taub-NUT metric in the form of Newman *et al.*,[5] let $B_0=l$ and $C=2m$, and express A_0 in terms of ϕ and l:

$$ds^2=-f^{-2}dt''^2+(2l)^2f^2\sigma_z{}^2+(t^2+l^2)(\sigma_x{}^2+\sigma_y{}^2) \tag{24}$$

with

$$f^2=2\left[\frac{mt''+l^2-\frac{1}{8}\phi^2}{t''^2+l^2}\right]-1.$$

When $\phi\neq0$, t' can be computed from (12), (20), and (21) in terms of t'':

$$dt'=2(B_0/B^2)dt''=2B_0dt''/(B_0{}^2+t''^2)$$

so that the electromagnetic fields (14) are given by

$$\begin{aligned}E&=2\phi B_0t''/(t''^2+B_0{}^2)^2\\ H&=\phi(B_0{}^2-t''^2)/(t''^2+B_0{}^2)^2.\end{aligned} \tag{25}$$

IV. CONCLUSION

Many features of our metric (24) are similar to those of the Taub-NUT solution. For a finite range of time, during which $f^2<0$, or $A^2>0$, i.e.,

$$-m-(m^2+l^2-\tfrac{1}{4}\phi^2)^{1/2}<t<-m+(m^2+l^2-\tfrac{1}{4}\phi^2)^{1/2}, \tag{26}$$

it represents a closed universe with electromagnetic fields given by (25). For times outside this range (or for all t'' if $4(m^2+l^2)<\phi^2$), t'' is a spacelike ("radial") coordinate and σ_z is timelike. In this region the solution is related to the asymptotically flat NUT space in somewhat the same way as the Reissner-Nordström solution for a charged "particle" is related to the Schwarzschild

TABLE I. The different solutions contained in the metric given here as specializations. The arrows point in directions of increasing generalization.

	$\phi=0$		$\phi\neq 0$
$l=0$	Schwarzschild	→	Reissner-Nordström
	↓		↓
$l\neq 0$	Taub-NUT	→	present paper

solution (see Table I). In particular, when $l=0(=B_0)$, Eq. (25) shows that $\mathbf{E}$ vanishes. Our solution then goes over into the Reissner-Nordström solution for a magnetic pole. By a "duality rotation" the roles of the electric and magnetic fields can be interchanged to obtain the more familiar form of the Reissner-Nordström solution, representing an electric charge.

In the case $B_0\neq 0$ ("charged NUT space") both $\mathbf{E}$ and $\mathbf{H}$ fields differ from zero, and no duality rotation can make the field purely electric or purely magnetic, since according to (25) the two fields vary differently with the "radial" coordinate t.

The time development of the solution in the region representing a closed universe is very similar to that of Taub's solution. If we choose $c=2m=0$ we obtain a time-symmetric solution; if we take $A_0=B_0=\frac{2}{3}\phi$, the initial geometry is that of an (undistorted) 3-sphere, both homogeneous and isotropic, in which a preferred direction is found only in the electromagnetic fields. (This is the case referred to in the Introduction). All the solutions develop geometrical anisotropy $(A\neq B)$ in time. Equation (22) shows that A^2 increases from zero to a maximum and decreases again to zero. The total space volume $V=\int g^{1/2}d^3x=\int AB^2d^3x$, measured on the invariant varieties $t=$ const (or $t''=$ const) shows a similar behavior: the universe expands and recontracts like the familiar Friedman models. However, neither the Taub-NUT solution nor the present generalization shows any geometrical singularities on the timelike surface on which $A=f=V=0$, which separates the closed universe and the outer, asymptotically flat region.[12] To show this, note that all the terms in (19) are finite on this surface, in particular

$$\dot{A}/A=(1/2B_0)dA/dt''<\infty\,.$$

The analytic continuation of the Taub-NUT metric across this surface has been discussed by Misner and Taub[13] and these results apply also to the present solution. Although no singularity exists in four-space, every spacelike hypersurface which is pushed forward beyond the region represented by (26) does become singular.[6]

Another generalization of Taub's solution, to the case of dust-filled universes, has been treated via numerical integration by C. Behr.[4] Whereas the solutions presented here may be said to correspond to universes of various ratios of electromagnetic to gravitational radiation content, Behr's solutions correspond to various ratios of dust to gravitational radiation content. All the latter solutions show geometric singularities, since particle paths must cross in a finite proper time,[14] leading to infinite mass-energy density, T_{00}. Thus Taub's solution and its generalizations give us a continuous family of solutions with the Friedmann universe as one limit and the Schwarzschild or Reissner-Nordström solution as the other limit.

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12 For the Taub-NUT case this was shown by V. Joseph, Proc. Cambridge Phil. Soc. **53**, 836 (1957).

13 C. W. Misner and A. H. Taub (to be published); also see C. W. Misner, J. Math. Phys. **4**, 934, Ref. 19 (1963).

14 A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955); and **106**, 172 (1957); A. Komar, *ibid.* **104**, 544 (1956).